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## **$S^3$ and $S^4$ Reductions of Type IIA Supergravity**

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### ABSTRACT

We construct a consistent reduction of type IIA supergravity on  $S^3$ , leading to a maximal gauged supergravity in seven dimensions with the full set of massless  $SO(4)$  Yang-Mills fields. We do this by starting with the known  $S^4$  reduction of eleven-dimensional supergravity, and showing that it is possible to take a singular limit of the resulting standard  $SO(5)$ -gauged maximal supergravity in seven dimensions, whose eleven-dimensional interpretation involves taking a limit where the internal 4-sphere degenerates to  $\mathbb{R} \times S^3$ . This allows us to reinterpret the limiting  $SO(4)$ -gauged theory in seven dimensions as the  $S^3$  reduction of type IIA supergravity. We also obtain the consistent  $S^4$  reduction of type IIA supergravity, which gives an  $SO(5)$ -gauged maximal supergravity in  $D = 6$ .

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# 1 Introduction

The study of Kaluza-Klein sphere reductions of supergravities has so far concentrated mostly on the examples where the theories admit vacuum solutions of the form  $\text{AdS} \times \text{Sphere}$ , which are the near-horizon structures of certain  $p$ -brane solutions of the theories.<sup>1</sup> These include 11-dimensional supergravity, which has  $\text{AdS}_4 \times S^7$  and  $\text{AdS}_7 \times S^4$  vacuum solutions, and type IIB supergravity, which has an  $\text{AdS}_5 \times S^5$  vacuum solution. The  $S^7$  and  $S^4$  reduction Ansätze for 11-dimensional supergravity were presented in [1, 2]. For type IIB supergravity the  $S^5$  reduction of its  $SL(2, \mathbb{R})$ -singlet subsector, which gives rise to five-dimensional fields including the entire set of  $SO(6)$  gauge bosons, was given in [3]. Explicit reduction Ansätze for various subsectors of these supergravity reductions can be found in [4, 5, 6, 7, 8, 9, 10, 11, 12].

In general, vacuum supergravity solutions with non-trivial field-strength fluxes are of the form of warped products of a certain spacetime geometry and internal spheres. The consistency of sphere reductions in such cases have been much less fully studied. The first example of this type was the consistent warped  $S^4$  reduction [13] of massive type IIA supergravity, to give rise to the massive  $SU(2)$ -gauged supergravity in  $D = 6$ . The vacuum  $\text{AdS}_6$  solution can be viewed as the near-horizon structure [14] of an intersecting D4-D8 brane [15].

Further examples of consistent sphere reductions were obtained in [16], where the resulting theories admit “vacuum solutions” that are domain walls rather than AdS spacetimes. In [16], a necessary condition for the consistency of a sphere reduction of a theory was given. Namely, if a theory can be consistently reduced on  $S^n$ , with a massless truncation that retains all the  $SO(n+1)$  Yang-Mills gauge fields, then a necessary requirement is that if instead a toroidal reduction on  $T^n$  is performed, this must give rise to the same content of massless fields.<sup>2</sup> Furthermore, the  $T^n$ -reduced theory must have at least an  $SO(n+1)$  global symmetry, with sufficiently many abelian vector fields to supply at least those of the adjoint representation of  $SO(n+1)$ . These conditions are very restrictive, and only in limited cases can a consistent sphere reduction that retains all the Yang-Mills fields occur.

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<sup>1</sup>Throughout this paper we are concerned only with those “remarkable” Kaluza-Klein sphere reductions for which no known group-theoretic argument guaranteeing the consistency of the reduction exists. Consistent reductions on  $S^3$ , or indeed any group manifold  $G$ , can always be performed in the case where one truncates to the sector of singlets under the right action of the group  $G$ , but the consistency in such a case is guaranteed, and therefore is not of interest to us in the present paper.

<sup>2</sup>This is because by turning off the gauge-coupling parameter  $g$ , by sending the radius of the  $n$ -sphere to infinity, we must recover the same massless field content as would result from a flat (toroidal) reduction.

In the type IIA and type IIB theories, the NS-NS branes and D-branes have near-horizon structures of the form (Domain wall)  $\times S^n$ , for various values of  $n$ . It is easy to verify in each case that if a Kaluza-Klein reduction of the theory on  $T^n$  is performed, this yields a scalar coset  $SL(n+1, \mathbb{R})/SO(n+1)$  in the lower dimension, since this theory can instead be obtained from a  $T^{n+1}$  reduction from  $D = 11$ . In particular, therefore, this theory has an  $SO(n+1)$  global symmetry subgroup. It was noted [17] that there exists a “dual frame” for each D-brane, in which the near-horizon structure of the brane generically becomes  $AdS_{10-n} \times S^n$ , while instead it becomes Minkowski  $\times S^3$  when  $n = 3$ . This leads to the conjectured Domain Wall/QFT correspondence [17], generalising the notion of the AdS/CFT correspondence [18, 19, 20]. This correspondence was generalised to lower dimensions in [21].

Further evidence for Domain Wall/QFT correspondence was obtained in [22], where it was shown that it is consistent to reduce to the subsector scalars associated with the Cartan generators in the scalar coset  $SL(n+1, \mathbb{R})/SO(n+1)$ . The multi-parameter domain-wall solutions of these lower-dimensional theories can then be lifted back to the higher dimension, where they correspond to certain ellipsoidal distributions of the  $p$ -branes, thus implying that these domain wall geometries correspond to the Coulomb branch of the quantum field theory, and generalising the results on the Coulomb branch in the AdS/CFT correspondence, discussed in [23, 24, 25, 26, 8, 27]. Interestingly, the wave equations for minimally-coupled scalar fluctuations in the lower-dimensional domain-wall backgrounds depend only on the dimension of the internal sphere used in the reduction [22].

The consistency of the Kaluza-Klein reduction in this Cartan subsector of scalar fields leads one to believe further that it is consistent to reduce the type IIA and type IIB theories on the relevant  $n$ -spheres, while retaining all the massless fields. For non-trivial vacuum NS-NS flux, both the type IIA and type IIB theories can be expected to be consistently reducible on  $S^3$  and on  $S^7$ , and indeed, for  $N = 1$  supergravity, the consistency has been demonstrated, and the corresponding gauged supergravities in  $D = 7$  and  $D = 3$  were obtained in [16]. For non-trivial vacuum R-R flux, we expect that it is consistent to reduce the type IIA theory on  $S^n$  with  $n = 2, 4, 6, 8$  and the type IIB theory on  $S^n$  with  $n = 1, 3, 5, 7$ . Indeed, the  $S^4$  and  $S^7$  reductions of type IIA, where  $S^4$  and  $S^7$  are associated with the D4-brane and NS-NS string, can be established from the  $S^1$  reduction of the corresponding  $S^4$  or  $S^7$  reduction of eleven-dimensional supergravity. In section 5, we carry this out explicitly for the  $S^4$  reduction of the type IIA theory.

First, we demonstrate explicitly that it is consistent to perform an  $S^3$  reduction of the

type IIA theory, while retaining all the massless fields, including in particular the entire set of  $SO(4)$  Yang-Mills gauge fields. This case is of particular interest because  $S^3$  is itself the group manifold  $SU(2)$ , and strings propagating in group-manifold backgrounds have been extensively studied in the past. It should be emphasised though that typically when Kaluza-Klein reductions on a group manifold  $G$  have been discussed in the literature, a truncation is performed in which only those fields that are singlets under the left action  $G_L$  of the  $G_L \times G_R$  isometry group are retained. For example, the  $S^3$  reduction of  $N = 1$  supergravity in  $D = 10$ , retaining only one  $SU(2)$  Yang-Mills fields, was performed to give rise to gauged simple supergravity in  $D = 7$  with a domain wall vacuum solution [28]. Such a truncation guarantees that a consistent reduction can be performed, but it fails to exploit the much more remarkable fact that in this  $S^3$  case a reduction that retains *all* the  $SO(4) \sim SU(2)_L \times SU(2)_R$  Yang-Mills fields, and not merely those of  $SU(2)_L$ , is possible. A further reason for wishing to include all the gauge fields of  $SO(4)$  is that only then do we obtain a seven-dimensional theory with maximal supersymmetry.

We obtain the consistent  $S^3$  reduction of type IIA supergravity by taking a singular limit of the  $S^4$  reduction of eleven-dimensional supergravity, in which the  $S^4$  degenerates to  $\mathbb{R} \times S^3$ . In order to do this, we begin in section 2 by reviewing the  $S^4$  reduction from  $D = 11$ , first obtained in [2]. By substituting this into the eleven-dimensional Bianchi identity and equation of motion for the 4-form, we obtain complete and explicit seven-dimensional equations of motion, and the Lagrangian that generates them. We also discuss the “ungaunging limit” in which the radius of the 4-sphere is sent to infinity. (An analogous limit was also considered in [29], in the context of the  $U(1)^2$  subgroup of  $SO(4)$ .) In particular, we clarify certain aspects of this limiting process, showing that the limit is smooth in the seven-dimensional equations of motion, but pathological at the level of the gauged-supergravity Lagrangian.

In section 3 we take a different singular limit of the seven-dimensional  $SO(5)$ -gauged supergravity, in which an  $SO(4)$  gauging remains. Again, this is a smooth limit of the equations of motion, but not of the gauged supergravity Lagrangian. In section 4 we apply this limiting procedure to the  $S^4$  Kaluza-Klein reduction Ansatz of eleven-dimensional supergravity, showing that it corresponds to a degeneration of the 4-sphere to  $\mathbb{R} \times S^3$ . The reduction can then be viewed as an initial reduction to give type IIA supergravity in  $D = 10$ , followed by a reduction on  $S^3$ . By this means, we arrive at the consistent  $S^3$  reduction Ansatz for type IIA supergravity.

In section 5 we construct instead the consistent  $S^4$  Kaluza-Klein reduction of type IIA

supergravity. This can again be obtained from the starting point of the  $S^4$  reduction of the eleven-dimensional theory. In this case we do not need to take any singular limit in the internal directions, but rather, we perform a standard Kaluza-Klein  $S^1$  reduction of the original seven-dimensional theory coming from  $D = 11$ , and show how this can be reinterpreted as an  $S^4$  reduction of type IIA supergravity. The paper ends with concluding remarks in section 6.

## 2 The $S^4$ reduction of eleven-dimensional supergravity

### 2.1 Metric and 4-form Ansatz

The complete Ansatz for the  $S^4$  reduction of eleven-dimensional supergravity was obtained in [2], using a formalism based on an analysis of the supersymmetry transformation rules. One may also study the reduction from a purely bosonic standpoint, by verifying that if the Ansatz is substituted into the eleven-dimensional equations of motion, it consistently yields the equations of motion of the seven-dimensional gauged  $SO(5)$  supergravity. We shall carry out this procedure here, in order to establish notation, and to obtain the complete system of seven-dimensional bosonic equations of motion, which we shall need in the later part of the paper.

After some manipulation, the Kaluza-Klein  $S^4$  reduction Ansatz obtained in [2] for eleven-dimensional supergravity can be expressed as follows:

$$d\hat{s}_{11}^2 = \Delta^{1/3} ds_7^2 + \frac{1}{g^2} \Delta^{-2/3} T_{ij}^{-1} D\mu^i D\mu^j, \quad (1)$$

$$\begin{aligned} \hat{F}_{(4)} &= \frac{1}{4!} \epsilon_{i_1 \dots i_5} \left[ -\frac{1}{g^3} U \Delta^{-2} \mu^{i_1} D\mu^{i_2} \wedge \dots \wedge D\mu^{i_5} \right. \\ &\quad + \frac{4}{g^3} \Delta^{-2} T^{i_1 m} DT^{i_2 n} \mu^m \mu^n D\mu^{i_3} \wedge \dots \wedge D\mu^{i_5} \\ &\quad \left. + \frac{6}{g^2} \Delta^{-1} F_{(2)}^{i_1 i_2} \wedge D\mu^{i_3} \wedge D\mu^{i_4} T^{i_5 j} \mu^j \right] - T_{ij} * S_{(3)}^i \mu^j + \frac{1}{g} S_{(3)}^i \wedge D\mu^i, \end{aligned} \quad (2)$$

where

$$\begin{aligned} U &\equiv 2T_{ij} T_{jk} \mu^i \mu^k - \Delta T_{ii}, \quad \Delta \equiv T_{ij} \mu^i \mu^j, \\ F_{(2)}^{ij} &\equiv dA_{(1)}^{ij} + gA_{(1)}^{ik} \wedge A_{(1)}^{kj}, \quad D\mu^i \equiv d\mu^i + gA_{(1)}^{ij} \mu^j, \\ DT_{ij} &\equiv dT_{ij} + gA_{(1)}^{ik} T_{kj} + gA_{(1)}^{jk} T_{ik}, \quad \mu^i \mu^i \equiv 1, \end{aligned} \quad (3)$$

where the symmetric matrix  $T_{ij}$ , which parameterises the scalar coset  $SL(6, \mathbb{R})/SO(6)$ , is unimodular.

## 2.2 Derivation of the seven-dimensional equations of motion

Consider first the Bianchi identity  $d\hat{F}_{(4)} = 0$ . Substituting (2) into this, we obtain the following seven-dimensional equations:

$$D(T_{ij} * S_{(3)}^j) = F_{(2)}^{ij} \wedge S_{(3)}^j, \quad (4)$$

$$H_{(4)}^i = g T_{ij} * S_{(3)}^j + \frac{1}{8} \epsilon_{ij_1 \dots j_4} F_{(2)}^{j_1 j_2} \wedge F_{(2)}^{j_3 j_4}, \quad (5)$$

where we define

$$H_{(4)}^i \equiv D S_{(3)}^i = d S_{(3)}^i + g A_{(1)}^{ij} \wedge S_{(3)}^j. \quad (6)$$

Next, we substitute the Ansatz into the  $D = 11$  field equation  $d * \hat{F}_{(4)} = \frac{1}{2} \hat{F}_{(4)} \wedge \hat{F}_{(4)}$ . To do this, we need the eleven-dimensional Hodge dual  $* \hat{F}_{(4)}$ , which we find is given by

$$\begin{aligned} * \hat{F}_{(4)} &= -g U \epsilon_{(7)} - \frac{1}{g} T_{ij}^{-1} * D T^{ik} \mu_k \wedge D \mu^j + \frac{1}{2g^2} T_{ik}^{-1} T_{j\ell}^{-1} * F_{(2)}^{ij} \wedge D \mu^k \wedge D \mu^\ell \\ &\quad + \frac{1}{g^4} \Delta^{-1} T_{ij} S_{(3)}^i \mu^j \wedge W - \frac{1}{6g^3} \Delta^{-1} \epsilon_{ij\ell_1 \ell_2 \ell_3} * S_{(3)}^m T_{im} T_{jk} \mu^k \wedge D \mu^{\ell_1} \wedge D \mu^{\ell_2} \wedge D \mu^{\ell_3}, \end{aligned} \quad (7)$$

where

$$W \equiv \frac{1}{24} \epsilon_{i_1 \dots i_5} \mu^{i_1} D \mu^{i_2} \wedge \dots \wedge D \mu^{i_5}. \quad (8)$$

The field equation for  $\hat{F}_{(4)}$  then implies

$$\begin{aligned} D \left( T_{ik}^{-1} T_{j\ell}^{-1} * F_{(2)}^{ij} \right) &= -2g T_{i[k}^{-1} * D T_{\ell]i} - \frac{1}{2g} \epsilon_{i_1 i_2 i_3 k\ell} F_2^{i_1 i_2} H_{(4)}^{i_3} \\ &\quad + \frac{3}{2g} \delta_{i_1 i_2 k\ell}^{j_1 j_2 j_3 j_4} F_{(2)}^{i_1 i_2} \wedge F_{(2)}^{j_1 j_2} \wedge F_{(2)}^{j_3 j_4} - S_{(3)}^k \wedge S_{(3)}^\ell. \end{aligned} \quad (9)$$

$$\begin{aligned} D \left( T_{ik}^{-1} * D(T_{kj}) \right) &= 2g^2 (2T_{ik} T_{kj} - T_{kk} T_{ij}) \epsilon_{(7)} + T_{im}^{-1} T_{k\ell}^{-1} * F_{(2)}^{m\ell} \wedge F_{(2)}^{kj} \\ &\quad + T_{jk} * S_{(3)}^k \wedge S_{(3)}^i - \frac{1}{5} \delta_{ij} \left[ 2g^2 (2T_{ik} T_{ik} - 2(T_{ii})^2) \epsilon_{(7)} \right. \\ &\quad \left. + T_{nm}^{-1} T_{k\ell}^{-1} * F_{(2)}^{m\ell} \wedge F_{(2)}^{kn} + T_{k\ell} * S_{(3)}^k \wedge S_{(3)}^\ell \right], \end{aligned} \quad (10)$$

for the Yang-Mills and scalar equations of motion in  $D = 7$ .<sup>3</sup>

We find that all the equations of motion can be derived from the following seven-dimensional Lagrangian

$$\begin{aligned} \mathcal{L}_7 &= R * \mathbb{1} - \frac{1}{4} T_{ij}^{-1} * D T_{jk} \wedge T_{k\ell}^{-1} D T_{\ell i} - \frac{1}{4} T_{ik}^{-1} T_{j\ell}^{-1} * F_{(2)}^{ij} \wedge F_{(2)}^{k\ell} - \frac{1}{2} T_{ij} * S_{(3)}^i \wedge S_{(3)}^j \\ &\quad + \frac{1}{2g} S_{(3)}^i \wedge H_{(4)}^i - \frac{1}{8g} \epsilon_{ij_1 \dots j_4} S_{(3)}^i \wedge F_{(2)}^{j_1 j_2} \wedge F_{(2)}^{j_3 j_4} + \frac{1}{g} \Omega_{(7)} - V * \mathbb{1}, \end{aligned} \quad (11)$$

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<sup>3</sup>Note from (9) that it would be inconsistent to set the Yang-Mills fields to zero while retaining the scalars  $T_{ij}$ , since the currents  $T_{i[k}^{-1} * D T_{\ell]i}$  act as sources for them. A truncation where the Yang-Mills fields are set to zero *is* consistent, however, if the scalars are also truncated to the diagonal subsector  $T_{ij} = \text{diag}(X_1, X_2, \dots, X_6)$ , as in the consistent reductions constructed in [8, 9].

where  $H_{(4)}^i$  are given by (6) and the potential  $V$  is given by

$$V = \frac{1}{2}g^2 \left( 2T_{ij} T_{ij} - (T_{ii})^2 \right), \quad (12)$$

and  $\Omega_{(7)}$  is a Chern-Simons type of term built from the Yang-Mills fields, which has the property that its variation with respect to  $A_{(1)}^{ij}$  gives

$$\delta\Omega_{(7)} = \frac{3}{4}\delta_{i_1 i_2 k \ell}^{j_1 j_2 j_3 j_4} F_{(2)}^{i_1 i_2} \wedge F_{(2)}^{j_1 j_2} \wedge F_{(2)}^{j_3 j_4} \wedge \delta A_{(1)}^{k \ell}. \quad (13)$$

Note that the  $S_{(3)}^i$  are viewed as fundamental fields in the Lagrangian, and that (5) is their first-order equation. In fact (11) is precisely the bosonic sector of the Lagrangian describing maximal gauged seven-dimensional supergravity that was derived in [30]. An explicit expression for the 7-form  $\Omega_{(7)}$  can be found there.

Although we have fully checked the eleven-dimensional Bianchi identity and field equation for  $\hat{F}_{(4)}$  here, we have not completed the task of substituting the Ansatz into the eleven-dimensional Einstein equations. This would be an extremely complicated calculation, on account of the Yang-Mills gauge fields. However, various complete consistency checks, including the higher-dimensional Einstein equation, have been performed in various truncations of the full  $N = 4$  maximal supergravity embedding, including the  $N = 2$  gauged theory in [5], and the non-supersymmetric truncation in [9] where the gauge fields are set to zero and only the diagonal scalars in  $T_{ij}$  are retained. All the evidence points to the full consistency of the reduction.<sup>4</sup>

It is perhaps worth making a few further remarks on the nature of the reduction Ansatz. One might wonder whether the Ansatz (2) on the 4-form field strength  $\hat{F}_{(4)}$  could be re-expressed as an Ansatz on its potential  $\hat{A}_{(3)}$ . As it stands, (2) only satisfies the Bianchi identity  $d\hat{F}_{(4)} = 0$  by virtue of the lower-dimensional equations (4) and (5). However, if (5) is substituted into (2), we obtain an expression that satisfies  $d\hat{F}_{(4)} = 0$  without the use of any lower-dimensional equations. However, one does still have to make use of the fact that the  $\mu^i$  coordinates satisfy the constraint  $\mu^i \mu^i = 1$ , and this prevents one from writing an explicit Ansatz for  $\hat{A}_{(3)}$  that has a manifest  $SO(5)$  symmetry. One could solve for one of the  $\mu^i$  in terms of the others, but this would break the *manifest* local symmetry from  $SO(5)$  to  $SO(4)$ . In principle though, this could be done, and then one could presumably substitute the resulting Ansatz directly into the eleven-dimensional Lagrangian. After integrating

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<sup>4</sup>The original demonstration in [2], based on the reduction of the eleven-dimensional supersymmetry transformation rules, also provides extremely compelling evidence. Strictly speaking the arguments presented there also fall short of a complete and rigorous proof, since they involve an approximation in which the quartic fermion terms in the theory are neglected.

out the internal 4-sphere directions, one could then in principle obtain a seven-dimensional Lagrangian in which, after re-organising terms, the local  $SO(5)$  symmetry could again become manifest.

It should, of course, be emphasised that merely substituting an Ansatz into a Lagrangian and integrating out the internal directions to obtain a lower-dimensional Lagrangian is justifiable only if one already has an independent proof of the consistency of the proposed reduction Ansatz.<sup>5</sup> If one is in any case going to work with the higher-dimensional field equations in order to prove the consistency, it is not clear that there would be any significant benefit to be derived from then re-expressing the Ansatz in a form where it could be substituted into the Lagrangian.

### 2.3 Ungauging: the $g \rightarrow 0$ limit

It is interesting to observe that one cannot take the limit  $g \rightarrow 0$  in the Lagrangian (11), on account of the terms proportional to  $g^{-1}$  in the second line. We know, on the other hand, that it must be possible to recover the ungauged  $D = 7$  theory by turning off the gauge coupling constant. In fact the problem is associated with a pathology in taking the limit at the level of the Lagrangian, rather than in the equations of motion. This can be seen by looking instead at the seven-dimensional equations of motion, which were given earlier. The only apparent obstacle to taking the limit  $g \rightarrow 0$  is in the Yang-Mills equations (9), but in fact this illusory. If we substitute the first-order equation (5) into (9) it gives

$$D\left(T_{ik}^{-1}T_{j\ell}^{-1}*F_{(2)}^{ij}\right) = -2gT_{i[k}^{-1}*DT_{\ell]i} - \frac{1}{2}\epsilon_{i_1 i_2 i_3 k\ell} F_2^{i_1 i_2} \wedge T_{ij} *S_{(3)}^j - S_{(3)}^k \wedge S_{(3)}^\ell, \quad (14)$$

which has a perfectly smooth  $g \rightarrow 0$  limit. It is clear that equations of motion (5) and (10) and the Einstein equations of motion also have a smooth limit. (The reason why the Einstein equations have the smooth limit is because the  $1/g$  terms in the Lagrangian (11) do not involve the metric, and thus they give no contribution.)

Unlike in the gauged theory, we should not treat the  $S_{(3)}^i$  fields as fundamental variables in a Lagrangian formulation in the ungauged limit. This is because once the gauge coupling  $g$  is sent to zero, the fields  $S_{(3)}^i$  behave like 3-form field strengths. This can be seen from

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<sup>5</sup>A classic illustration is provided by the example of 5-dimensional pure gravity with an (inconsistent) Kaluza-Klein reduction in which the scalar dilaton is omitted. Substituting this into the 5-dimensional Einstein-Hilbert action yields the perfectly self-consistent Einstein-Maxwell action in  $D = 4$ , but fails to reveal that setting the scalar to zero is inconsistent with the internal component of the 5-dimensional Einstein equation.

the first-order equation of motion (5), which in the limit  $g \rightarrow 0$  becomes

$$dS_{(3)}^i = \frac{1}{8}\epsilon_{ij_1\cdots j_4} dA_{(1)}^{j_1 j_2} \wedge dA_{(1)}^{j_3 j_4}, \quad (15)$$

and should now be interpreted as a Bianchi identity. This can be solved by introducing 2-form gauge potentials  $A_{(2)}^i$ , with the  $S_{(3)}^i$  given by

$$S_{(3)}^i = dA_{(2)}^i + \frac{1}{8}\epsilon_{ij_1\cdots j_4} A_{(1)}^{j_1 j_2} \wedge dA_{(1)}^{j_3 j_4}. \quad (16)$$

In terms of these 2-form potentials, the equations of motion can now be obtained from the Lagrangian

$$\begin{aligned} \mathcal{L}_7^0 = & R * \mathbb{1} - \frac{1}{4}T_{ij}^{-1} * dT_{jk} \wedge T_{kl}^{-1} dT_{\ell i} - \frac{1}{4}T_{ik}^{-1} T_{j\ell}^{-1} * F_{(2)}^{ij} \wedge F_{(2)}^{k\ell} - \frac{1}{2}T_{ij} * S_{(3)}^i \wedge S_{(3)}^j \\ & + \frac{1}{2}A_{(1)}^{ij} \wedge S_{(3)}^i \wedge S_{(3)}^j - 2S_{(3)}^i \wedge A_{(2)}^j \wedge dA_{(1)}^{ij}, \end{aligned} \quad (17)$$

where  $S_{(3)}^i$  is given by (16). This is precisely the bosonic Lagrangian of the ungauged maximal supergravity in  $D = 7$ .

It is worth exploring in a little more detail why it is possible to take a smooth  $g \rightarrow 0$  limit in the seven-dimensional equations of motion, but not in the Lagrangian. We note that in this limit the Lagrangian (11) can be expressed as

$$\mathcal{L}_7 = \frac{1}{g}L + \mathcal{O}(1), \quad (18)$$

where

$$L = \frac{1}{2}S_{(3)}^i dS_{(3)}^i - \frac{1}{8}\epsilon_{ij_1\cdots j_4} S_{(3)}^i \wedge F_{(2)}^{j_1 j_2} \wedge F_{(2)}^{j_3 j_4} + \Omega_{(7)}. \quad (19)$$

The term  $L/g$ , which diverges in the  $g \rightarrow 0$  limit, clearly emphasises that the Lagrangian (17) is not merely the  $g \rightarrow 0$  limit of (11). However if we make use of the equations of motion, we find that in the  $g \rightarrow 0$  limit the  $S_{(3)}^i$  can be solved by (16). Substituting this into (19), we find that in this limit it becomes

$$L = \frac{1}{16}\epsilon_{ij_1\cdots j_4} dA_{(2)}^i \wedge dA_{(1)}^{j_1 j_2} \wedge dA_{(1)}^{j_3 j_4} + \mathcal{O}(g), \quad (20)$$

and so the singular terms in  $L/g$  form a total derivative and hence can be subtracted from the Lagrangian. This analysis explains why it is possible to take a smooth  $g \rightarrow 0$  limit in the equations of motion, but not in the Lagrangian.

### 3 The gauged $SO(4)$ limit of maximal $D = 7$ supergravity

Here we examine, at the level of the seven-dimensional theory itself, how to take a limit in which the  $SO(5)$  gauged sector is broken down to  $SO(4)$ . In a later section, we shall

show how this can be interpreted as an  $S^3$  reduction of type IIA supergravity. We shall do that by showing how to take a limit in which the internal  $S^4$  in the original reduction from  $D = 11$  becomes  $\mathbb{R} \times S^3$ . For now, however, we shall examine the  $SO(4)$ -gauged limit entirely from the perspective of the seven-dimensional theory itself.

To take the limit, we break the  $SO(5)$  covariance by splitting the  $\underline{5}$  index  $i$  as

$$i = (0, \alpha), \quad (21)$$

where  $1 \leq \alpha \leq 4$ . We also introduce a constant parameter  $\lambda$ , which will be sent to zero as the limit is taken. We find that the various seven-dimensional fields, and the  $SO(5)$  gauge-coupling constant, should be scaled as follows:

$$\begin{aligned} g &= \lambda^2 \tilde{g}, & A_{(1)}^{0\alpha} &= \lambda^3 \tilde{A}_{(1)}^{0\alpha}, & A_{(1)}^{\alpha\beta} &= \lambda^{-2} \tilde{A}_{(1)}^{\alpha\beta}, \\ S_{(3)}^0 &= \lambda^{-4} \tilde{S}_{(3)}^0, & S_{(3)}^\alpha &= \lambda \tilde{S}_{(3)}^\alpha. \end{aligned} \quad (22)$$

$$T_{ij}^{-1} = \begin{pmatrix} \lambda^{-8} \Phi & \lambda^{-3} \Phi \chi_\alpha \\ \lambda^{-3} \Phi \chi_\alpha & \lambda^2 M_{\alpha\beta}^{-1} + \lambda^2 \Phi \chi_\alpha \chi_\beta \end{pmatrix}.$$

As we show in the next section, this rescaling corresponds to a degeneration of  $S^4$  to  $R \times S^3$ . Note that in this rescaling, we have also performed a decomposition of the scalar matrix  $T_{ij}^{-1}$  that is of the form of a Kaluza-Klein metric decomposition. It is useful also to present the consequent decomposition for  $T_{ij}$ , which turns out to be

$$T_{ij} = \begin{pmatrix} \lambda^8 \Phi^{-1} + \lambda^8 \chi_\gamma \chi^\gamma & -\lambda^3 \chi^\alpha \\ -\lambda^3 \chi^\alpha & \lambda^{-2} M_{\alpha\beta} \end{pmatrix}. \quad (23)$$

Calculating the determinant, we get

$$\det(T_{ij}) = \Phi^{-1} \det(M_{\alpha\beta}). \quad (24)$$

Since we know that  $\det(T_{ij}) = 1$ , it follows that

$$\Phi = \det(M_{\alpha\beta}). \quad (25)$$

The fields  $\chi_\alpha$  are “axionic” scalars. Note that we shall also have

$$\begin{aligned} H_{(4)}^0 &= \lambda^{-4} \tilde{H}_{(4)}^0, & H_{(4)}^\alpha &= \lambda \tilde{H}_{(4)}^\alpha, \\ \tilde{H}_{(4)}^0 &= d\tilde{S}_{(3)}^0, & \tilde{H}_{(4)}^\alpha &= \tilde{D}\tilde{S}_{(3)}^\alpha - \tilde{g} \tilde{A}_{(1)}^{0\alpha} \wedge \tilde{S}_{(3)}^0. \end{aligned} \quad (26)$$

We have defined an  $SO(4)$ -covariant exterior derivative  $\tilde{D}$ , which acts on quantities with  $SO(4)$  indices  $\alpha, \beta, \dots$  in the obvious way:

$$\tilde{D} X_\alpha = dX_\alpha + \tilde{g} \tilde{A}_{(1)}^{a\beta} X_\beta, \quad (27)$$

etc. It is helpful also to make the following further field redefinitions:

$$\begin{aligned} G_{(2)}^\alpha &\equiv \tilde{F}_{(2)}^{0\alpha} + \chi_\beta \tilde{F}_{(2)}^{\beta\alpha}, \\ G_{(3)}^\alpha &\equiv \tilde{S}_{(3)}^\alpha - \chi_\alpha \tilde{S}_{(3)}^0, \\ G_{(1)}^\alpha &\equiv \tilde{D}\chi_\alpha - \tilde{g} \tilde{A}_{(1)}^{0\alpha}, \end{aligned} \quad (28)$$

where  $\tilde{F}_{(2)}^{0\alpha} \equiv \tilde{D}\tilde{A}_{(1)}^{0\alpha}$ .

We may now substitute these redefined fields into the seven-dimensional equations of motion. We find that a smooth limit in which  $\lambda$  is sent to zero exists, leading to an  $SO(4)$ -gauged seven-dimensional theory. Our results for the seven-dimensional equations of motion are as follows. The fields  $H_{(4)}^i$  become

$$\tilde{H}_{(4)}^0 = d\tilde{S}_{(3)}^0, \quad \tilde{H}_{(4)}^\alpha = \tilde{D}G_{(3)}^\alpha + G_{(1)}^\alpha \wedge \tilde{S}_{(3)}^0 + \chi_\alpha dS_{(3)}^0. \quad (29)$$

The first-order equations (6) give

$$\begin{aligned} \tilde{H}_{(4)}^0 &= \frac{1}{8}\epsilon_{\alpha_1\cdots\alpha_4} \tilde{F}_{(2)}^{\alpha_1\alpha_2} \wedge \tilde{F}_{(2)}^{\alpha_3\alpha_4}, \\ \tilde{F}_{(4)}^\alpha &= \tilde{g} M_{\alpha\beta} *G_{(3)}^\beta - \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} G_{(2)}^\beta \wedge \tilde{F}_{(2)}^{\gamma\delta} - G_{(1)}^\alpha \wedge \tilde{S}_{(3)}^0, \end{aligned} \quad (30)$$

where we have defined

$$F_{(4)}^\alpha \equiv \tilde{D}G_{(3)}^\alpha. \quad (31)$$

The second-order equations (4) (which are nothing but Bianchi identities following from (6)) become

$$\begin{aligned} d(\Phi^{-1} * \tilde{S}_{(3)}^0) &= M_{\alpha\beta} *G_{(3)}^\alpha \wedge G_{(1)}^\beta + G_{(2)}^\alpha \wedge G_{(3)}^\alpha, \\ \tilde{D}(M_{\alpha\beta} *G_{(3)}^\beta) &= \tilde{F}_{(2)}^{\alpha\beta} \wedge G_{(3)}^\beta - G_{(2)}^\alpha \wedge \tilde{S}_{(3)}^0. \end{aligned} \quad (32)$$

The Yang-Mills equations (9) become

$$\begin{aligned} \tilde{D}(\Phi M_{\alpha\beta}^{-1} *G_{(2)}^\beta) &= \tilde{g} \Phi M_{\alpha\beta} *G_{(1)}^\beta - \tilde{S}_{(3)}^0 \wedge G_{(3)}^\alpha - \frac{1}{2}\epsilon_{\alpha\beta_1\beta_2\beta_3} M_{\beta_3\gamma} \tilde{F}_{(2)}^{\beta_1\beta_2} \wedge *G_{(3)}^\gamma, \\ \tilde{D}[M_{\gamma\alpha}^{-1} M_{\delta\beta}^{-1} * \tilde{F}_{(2)}^{\gamma\delta}] &= -2\tilde{g} M_{\gamma[\alpha}^{-1} * \tilde{D}M_{\beta]\gamma} - G_{(3)}^\alpha \wedge G_{(3)}^\beta + \Phi M_{\alpha\gamma}^{-1} G_{(1)}^\beta \wedge *G_{(2)}^\gamma \\ &\quad - \Phi M_{\beta\gamma}^{-1} G_{(1)}^\alpha \wedge *G_{(2)}^\gamma - \epsilon_{\alpha\beta\gamma\delta} M_{\delta\lambda} G_{(2)}^\gamma \wedge *G_{(3)}^\lambda - \frac{1}{2}\Phi^{-1} \epsilon_{\alpha\beta\gamma\delta} \tilde{F}_{(2)}^{\gamma\delta} \wedge *\tilde{S}_{(3)}^0. \end{aligned} \quad (33)$$

Finally, the scalar field equations (10) give the following:

$$\begin{aligned} d(\Phi^{-1} *d\Phi) &= \Phi M_{\alpha\beta} *G_{(1)}^\alpha \wedge G_{(1)}^\beta + \Phi M_{\alpha\beta}^{-1} *G_{(2)}^\alpha \wedge G_{(2)}^\beta - \Phi^{-1} *\tilde{S}_{(3)}^0 \wedge \tilde{S}_{(3)}^0 + \frac{1}{5}Q, \\ \tilde{D}(\Phi M_{\alpha\beta} *G_{(1)}^\beta) &= \Phi M_{\beta\gamma}^{-1} *G_{(2)}^\gamma \wedge \tilde{F}_{(2)}^{\alpha\beta} - M_{\alpha\beta} *G_{(3)}^\beta \wedge \tilde{S}_{(3)}^0, \\ \tilde{D}(M_{\alpha\gamma}^{-1} * \tilde{D}M_{\gamma\beta}) &= \Phi M_{\beta\gamma} *G_{(1)}^\gamma \wedge G_{(1)}^\alpha + M_{\beta\gamma} *G_{(3)}^\gamma \wedge G_{(3)}^\alpha - \Phi M_{\alpha\gamma}^{-1} *G_{(2)}^\gamma \wedge G_{(2)}^\beta \\ &\quad + M_{\alpha\gamma}^{-1} M_{\lambda\delta}^{-1} * \tilde{F}_{(2)}^{\gamma\delta} \wedge \tilde{F}_{(2)}^{\lambda\beta} + 2\tilde{g}^2 (2M_{\alpha\gamma} M_{\gamma\beta} - M_{\gamma\gamma} M_{\alpha\beta}) \epsilon_{(7)} - \frac{1}{5}\delta_{\alpha\beta} Q. \end{aligned} \quad (34)$$

In these equations, the quantity  $Q$  is the limit of the trace term multiplying  $\delta_{ij}$  in (10), and is given by

$$\begin{aligned} Q = & 2\tilde{g}^2 \left( 2M_{\alpha\beta} M_{\alpha\beta} - (M_{\alpha\alpha})^2 \right) \epsilon_{(7)} - M_{\alpha\gamma}^{-1} M_{\beta\delta}^{-1} * \tilde{F}_{(2)}^{\alpha\beta} \wedge \tilde{F}_{(2)}^{\gamma\delta} \\ & + \Phi^{-1} * \tilde{S}_{(3)}^0 \wedge \tilde{S}_{(3)}^0 - 2\Phi M_{\alpha\beta}^{-1} * G_{(2)}^\alpha \wedge G_{(2)}^\beta + M_{\alpha\beta} * G_{(3)}^\alpha \wedge G_{(3)}^\beta. \end{aligned} \quad (35)$$

Having obtained the seven-dimensional equations of motion for the  $SO(4)$ -gauged limit, we can now seek a Lagrangian from which they can be generated. A crucial point is that the equations involving  $\tilde{H}_{(4)}^0$  in (29) and (30) give

$$d\tilde{S}_{(3)}^0 = \frac{1}{8}\epsilon_{\alpha_1\cdots\alpha_4} \tilde{F}_{(2)}^{\alpha_1\alpha_2} \wedge \tilde{F}_{(2)}^{\alpha_3\alpha_4}, \quad (36)$$

which allows us to strip off the exterior derivative by writing

$$\tilde{S}_{(3)}^0 = dA_{(2)} + \omega_{(3)}, \quad (37)$$

where  $\tilde{S}_{(3)}^0$  is now viewed as a field strength with 2-form potential  $A_{(2)}$ , and

$$\omega_{(3)} \equiv \frac{1}{8}\epsilon_{\alpha_1\cdots\alpha_4} (\tilde{F}_{(2)}^{\alpha_1\alpha_2} \wedge \tilde{A}_{(1)}^{\alpha_3\alpha_4} - \frac{1}{3}\tilde{g} \tilde{A}_{(1)}^{\alpha_1\alpha_2} \wedge \tilde{A}_{(1)}^{\alpha_3\beta} \wedge \tilde{A}_{(1)}^{\beta\alpha_4}). \quad (38)$$

We can now see that the equations of motion can be derived from the following seven-dimensional Lagrangian, in which  $A_{(2)}$ , and not its field strength  $\tilde{S}_{(3)}^0 \equiv dA_{(2)} + \omega_{(3)}$ , is viewed as a fundamental field:

$$\begin{aligned} \mathcal{L}_7 = & R * \mathbb{1} - \frac{5}{16}\Phi^{-2} * d\Phi \wedge d\Phi - \frac{1}{4}M_{\alpha\beta}^{-1} * \tilde{D}M_{\beta\gamma} \wedge M_{\gamma\delta}^{-1} \tilde{D}M_{\delta\alpha} - \frac{1}{2}\Phi^{-1} * \tilde{S}_{(3)}^0 \wedge \tilde{S}_{(3)}^0 \\ & - \frac{1}{4}M_{\alpha\gamma}^{-1} M_{\beta\delta}^{-1} * \tilde{F}_{(2)}^{\alpha\beta} \wedge \tilde{F}_{(2)}^{\gamma\delta} - \frac{1}{2}\Phi M_{\alpha\beta}^{-1} * G_{(2)}^\alpha \wedge G_{(2)}^\beta - \frac{1}{2}\Phi M_{\alpha\beta} * G_{(1)}^\alpha \wedge G_{(1)}^\beta \\ & - \frac{1}{2}M_{\alpha\beta} * G_{(3)}^\alpha \wedge G_{(3)}^\beta - \tilde{V} * \mathbb{1} + \frac{1}{2\tilde{g}}\tilde{D}\tilde{S}_{(3)}^\alpha \wedge \tilde{S}_{(3)}^\alpha + \tilde{S}_{(3)}^\alpha \wedge \tilde{S}_{(3)}^0 \wedge A_{(1)}^{0\alpha} \\ & + \frac{1}{2\tilde{g}}\epsilon_{\alpha\beta\gamma\delta} \tilde{S}_{(3)}^\alpha \wedge \tilde{F}_{(2)}^{0\beta} \wedge \tilde{F}_{(2)}^{\gamma\delta} + \frac{1}{4}\epsilon_{\alpha_1\cdots\alpha_4} \tilde{S}_{(3)}^0 \wedge \tilde{F}_{(2)}^{\alpha_1\alpha_2} \wedge \tilde{A}_{(1)}^{0\alpha_3} \wedge \tilde{A}_{(1)}^{0\alpha_4} + \frac{1}{\tilde{g}}\tilde{\Omega}_{(7)}, \end{aligned} \quad (39)$$

where  $\tilde{\Omega}_{(7)}$  is built purely from  $\tilde{A}_{(1)}^{\alpha\beta}$  and  $\tilde{A}_{(1)}^{0\alpha}$ . It is defined by the requirement that its variations with respect to  $\tilde{A}_{(1)}^{\alpha\beta}$  and  $\tilde{A}_{(1)}^{0\alpha}$  should produce the necessary terms in the equations of motion for these fields. Since it has a rather complicated structure, we shall not present it here. Note that the fields that are treated as fundamental in this Lagrangian are the metric and the scalars  $(\Phi, M_{\alpha\beta})$ , together with  $(\chi_\alpha, \tilde{A}_{(1)}^{\alpha\beta}, \tilde{A}_{(1)}^{0\alpha}, \tilde{S}_{(3)}^\alpha, A_{(2)})$ , but it should be borne in mind that  $\Phi$  is not independent of  $M_{\alpha\beta}$ , because of the relation (25). It can be useful, therefore, to define the unimodular matrix  $\tilde{M}_{\alpha\beta} \equiv \Phi^{-1/4} M_{\alpha\beta}$ , so that  $\tilde{M}_{\alpha\beta}$  and  $\Phi$  are independent fields. The scalar part of the Lagrangian (39) then becomes

$$\mathcal{L}_{\text{scal}} = -\frac{1}{4}\Phi^{-2} * d\Phi \wedge d\Phi - \frac{1}{4}\tilde{M}_{\alpha\beta}^{-1} * \tilde{D}\tilde{M}_{\beta\gamma} \wedge \tilde{M}_{\gamma\delta}^{-1} \tilde{D}\tilde{M}_{\delta\alpha}. \quad (40)$$

## 4 $\mathbb{R} \times S^3$ limit of the $S^4$ reduction

In the previous section, we obtained a scaling limit of the gauged  $SO(5)$  theory in seven dimensions, in which an  $SO(4)$  gauging survives. In this section, we apply this scaling procedure to the  $S^4$  reduction Ansatz of section 2, showing that it leads to a degeneration in which the 4-sphere becomes  $\mathbb{R} \times S^3$ . We can then reinterpret the reduction from  $D = 11$  as an initial “ordinary” Kaluza-Klein reduction step from  $D = 11$  to give the type IIA supergravity in  $D = 10$ , followed by a non-trivial reduction of the type IIA theory on  $S^3$ , in which the entire  $SO(4)$  isometry group is gauged.<sup>6</sup>

### 4.1 The $\mathbb{R} \times S^3$ reduction Ansatz

To take this limit, we combine the scalings of seven-dimensional quantities derived in the previous section with an appropriately-matched rescaling of the coordinates  $\mu^i$  defined on the internal 4-sphere. As in [29], we see that after splitting the  $\mu^i$  into  $\mu^0$  and  $\mu^\alpha$ , these additional scalings should take the form

$$\mu^0 = \lambda^5 \tilde{\mu}^0, \quad \mu^\alpha = \tilde{\mu}^\alpha. \quad (41)$$

In the limit where  $\lambda$  goes to zero, we see that the original constraint  $\mu^i \mu^i = 1$  becomes

$$\tilde{\mu}^\alpha \tilde{\mu}^\alpha = 1, \quad (42)$$

implying that the  $\tilde{\mu}^\alpha$  coordinates define a 3-sphere, while the coordinate  $\tilde{\mu}^0$  is now unconstrained and ranges over the real line  $\mathbb{R}$ .

Combining this with the rescalings of the previous section, we find that the  $S^4$  metric reduction Ansatz (1) becomes

$$d\hat{s}_{11}^2 = \lambda^{-2/3} \left[ \tilde{\Delta}^{1/3} ds_7^2 + \frac{1}{\tilde{g}^2} \tilde{\Delta}^{-2/3} M_{\alpha\beta}^{-1} \tilde{D}\tilde{\mu}^\alpha \tilde{D}\tilde{\mu}^\beta + \frac{1}{\tilde{g}^2} \tilde{\Delta}^{-2/3} \Phi (d\tilde{\mu}_0 + \tilde{g} \tilde{A}_{(1)}^{0\alpha} \tilde{\mu}^\alpha + \chi_\alpha \tilde{D}\tilde{\mu}^\alpha)^2 \right], \quad (43)$$

---

<sup>6</sup>The  $S^3$  reduction of type IIA supergravity discussed in [28], giving a seven-dimensional theory with just an  $SU(2)$  gauging, was rederived in [31] as a singular limit of the  $S^4$  reduction of  $D = 11$  supergravity that was obtained in [2]. Since the  $S^3$  reduction in [28] retains only the left-acting  $SU(2)$  of the  $SO(4) \sim SU(2)_L \times SU(2)_R$  of gauge fields, the consistency of that reduction is guaranteed by group-theoretic arguments, based on the fact that all the retained fields are singlets under the right-acting  $SU(2)_R$ . The subtleties of the consistency of the  $S^4$  reduction in [2] are therefore lost in the singular limit to  $\mathbb{R} \times S^3$  discussed in [31], since a truncation to the  $SU(2)_L$  subgroup of the  $SO(4)$  gauge group is made. By contrast, the  $\mathbb{R} \times S^3$  singular limit that we consider here retains all the fields of the  $S^4$  reduction in [2], and the proof of the consistency of the resulting  $S^3$  reduction of the type IIA theory follows from the non-trivial consistency of the reduction in [2], and has no simple group-theoretic explanation.

where

$$\tilde{\Delta} \equiv M_{\alpha\beta} \tilde{\mu}^\alpha \tilde{\mu}^\beta. \quad (44)$$

Thus  $\tilde{\mu}_0$  can be interpreted as the “extra” coordinate of a standard type of Kaluza-Klein reduction from  $D = 11$  to  $D = 10$ , with

$$ds_{11}^2 = e^{-\frac{1}{6}\phi} ds_{10}^2 + e^{\frac{4}{3}\phi} (d\tilde{\mu}_0 + \mathcal{A}_{(1)})^2. \quad (45)$$

By comparing (45) with (43), we can read off the  $S^3$  reduction Ansatz for the ten-dimensional fields. Thus we find that the ten-dimensional metric is reduced according to

$$ds_{10}^2 = \Phi^{1/8} \left[ \tilde{\Delta}^{1/4} ds_7^2 + \frac{1}{\tilde{g}^2} \tilde{\Delta}^{-3/4} M_{\alpha\beta}^{-1} \tilde{D}\tilde{\mu}^\alpha \tilde{D}\tilde{\mu}^\beta \right], \quad (46)$$

while the Ansatz for the dilaton  $\phi$  of the ten-dimensional theory is

$$e^{2\phi} = \tilde{\Delta}^{-1} \Phi^{3/2}. \quad (47)$$

Finally, the reduction Ansatz for the ten-dimensional Kaluza-Klein vector is

$$\mathcal{A}_{(1)} = \tilde{g} \tilde{A}_{(1)}^{0\alpha} \tilde{\mu}^\alpha + \chi_\alpha \tilde{D}\tilde{\mu}^\alpha. \quad (48)$$

These results for the  $S^3$  reduction of the ten-dimensional metric and dilaton agree precisely with the results obtained in [16]. (Note that the field  $\Phi$  is called  $Y$  there, and our  $M_{\alpha\beta}$  is called  $T_{ij}$  there.) Note that the field strength  $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$  following from (48) has the simple expression

$$\mathcal{F}_{(2)} = \tilde{g} G_{(2)}^\alpha \tilde{\mu}^\alpha + G_{(1)}^\alpha \wedge \tilde{D}\tilde{\mu}^\alpha. \quad (49)$$

So far, we have read off the reduction Ansätze for those fields of ten-dimensional type IIA supergravity that come from the reduction of the eleven-dimensional metric. The remaining type IIA fields come from the reduction of the eleven-dimensional 4-form. Under the standard Kaluza-Klein procedure, this reduces as follows:

$$\hat{F}_{(4)} = F_{(4)} + F_{(3)} \wedge (d\tilde{\mu}^0 + \mathcal{A}_{(1)}). \quad (50)$$

By applying the  $\lambda$ -rescaling derived previously to the  $S^4$  reduction Ansatz (2) for the eleven-dimensional 4-form, and comparing with (50), we obtain the following expressions for the  $S^3$  reduction Ansätze for the ten-dimensional 4-form and 3-form fields:

$$\begin{aligned} F_{(4)} &= \frac{\tilde{\Delta}^{-1}}{\tilde{g}^3} M_{\alpha\beta} G_{(1)}^\alpha \tilde{\mu}^\beta \wedge \widetilde{W} + \frac{\tilde{\Delta}^{-1}}{2\tilde{g}^2} \epsilon_{\alpha_1\dots\alpha_4} M_{\alpha_4\beta} \tilde{\mu}^\beta G_{(2)}^{\alpha_1} \wedge \tilde{D}\tilde{\mu}^{\alpha_2} \wedge \tilde{D}\tilde{\mu}^{\alpha_3} \\ &\quad - M_{\alpha\beta} * G_{(3)}^\alpha \tilde{\mu}^\beta + \frac{1}{\tilde{g}} G_{(3)}^\alpha \wedge \tilde{D}\tilde{\mu}^\alpha, \end{aligned} \quad (51)$$

$$\begin{aligned} F_{(3)} &= -\frac{\tilde{U}\tilde{\Delta}^{-2}}{\tilde{g}^3} \widetilde{W} + \frac{\tilde{\Delta}^{-2}}{2\tilde{g}^2} \epsilon_{\alpha_1\dots\alpha_4} M_{\alpha_1\beta} \tilde{\mu}^\beta \tilde{D}M_{\alpha_2\gamma} \tilde{\mu}^\gamma \wedge \tilde{D}\tilde{\mu}^{\alpha_3} \wedge \tilde{D}\tilde{\mu}^{\alpha_4} \\ &\quad + \frac{\tilde{\Delta}^{-1}}{2\tilde{g}^2} \epsilon_{\alpha_1\dots\alpha_4} M_{\alpha_1\beta} \tilde{\mu}^\beta \tilde{F}_{(2)}^{\alpha_2\alpha_3} \wedge \tilde{D}\tilde{\mu}^{\alpha_4} + \frac{1}{\tilde{g}} \tilde{S}_{(3)}^0, \end{aligned} \quad (52)$$

where

$$\widetilde{W} \equiv \frac{1}{6} \epsilon_{\alpha_1 \dots \alpha_4} \tilde{\mu}^{\alpha_1} \tilde{D}\tilde{\mu}^{\alpha_2} \wedge \tilde{D}\tilde{\mu}^{\alpha_3} \wedge \tilde{D}\tilde{\mu}^{\alpha_4}. \quad (53)$$

It is also useful to present the expressions for the ten-dimensional Hodge duals of the field strengths:

$$\begin{aligned} e^{\frac{1}{2}\phi} \bar{*} F_{(4)} &= \frac{1}{\tilde{g}} \Phi M_{\alpha\beta} * G_{(1)}^\alpha \tilde{\mu}^\beta - \frac{1}{\tilde{g}^2} \Phi M_{\alpha\beta}^{-1} * G_{(2)}^\alpha \wedge \tilde{D}\tilde{\mu}^\beta + \frac{\tilde{\Delta}^{-1}}{\tilde{g}^4} M_{\alpha\beta} G_{(3)}^\alpha \tilde{\mu}^\beta \wedge \widetilde{W} \\ &\quad + \frac{\tilde{\Delta}^{-1}}{2\tilde{g}^3} \epsilon_{\alpha_1 \dots \alpha_4} M_{\alpha_1\beta} \tilde{\mu}^\beta M_{\alpha_2\gamma} * G_{(3)}^\gamma \wedge \tilde{D}\tilde{\mu}^{\alpha_3} \wedge \tilde{D}\tilde{\mu}^{\alpha_4}, \\ e^{-\phi} \bar{*} F_{(3)} &= -\tilde{g} \widetilde{U} \epsilon_{(7)} - \frac{1}{\tilde{g}^3} \Phi^{-1} * \widetilde{S}_{(3)}^0 \wedge \widetilde{W} \\ &\quad + \frac{1}{2\tilde{g}^2} M_{\alpha\gamma}^{-1} M_{\beta\delta}^{-1} * \widetilde{F}_{(2)}^{\alpha\beta} \wedge \tilde{D}\tilde{\mu}^\gamma \wedge \tilde{D}\tilde{\mu}^\delta - \frac{1}{\tilde{g}} M_{\alpha\beta}^{-1} * \tilde{D}M_{\alpha\gamma} \tilde{\mu}^\gamma \wedge \tilde{D}\tilde{\mu}^\beta, \\ e^{\frac{3}{2}\phi} \bar{*} \mathcal{F}_{(2)} &= \frac{\tilde{\Delta}^{-1}\Phi}{\tilde{g}^5} * G_{(2)}^\alpha \tilde{\mu}^\alpha \wedge \widetilde{W} + \frac{\tilde{\Delta}^{-1}\Phi}{2\tilde{g}^4} \epsilon_{\alpha_1 \dots \alpha_4} M_{\alpha_1\beta} \tilde{\mu}^\beta M_{\alpha_2\gamma} * G_{(1)}^\gamma \wedge \tilde{D}\tilde{\mu}^{\alpha_3} \wedge \tilde{D}\tilde{\mu}^{\alpha_4}. \end{aligned} \quad (54)$$

(Here we are using  $\bar{*}$  to denote a Hodge dualisation in the ten-dimensional metric  $ds_{10}^2$ , to distinguish it from  $*$  which denotes the seven-dimensional Hodge dual in the metric  $ds_7^2$ . )

## 4.2 Verification of the reduction Ansatz

The consistency of the  $S^3$  reduction of the type IIA theory using the Ansatz that we obtained in the previous subsection is guaranteed by virtue of the consistency of the  $S^4$  reduction from  $D = 11$ . It is still useful, however, to examine the reduction directly, by substituting the Ansatz into the equations of motion of type IIA supergravity. By this means we can obtain an explicit verification of the validity of the limiting procedures that we applied in obtaining the  $S^3$  reduction Ansatz.

The bosonic Lagrangian for type IIA supergravity can be written as

$$\begin{aligned} \mathcal{L}_{10} &= R \bar{*} \mathbb{1} - \frac{1}{2} \bar{*} d\phi \wedge d\phi - \frac{1}{2} e^{\frac{3}{2}\phi} \bar{*} \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} - \frac{1}{2} e^{\frac{1}{2}\phi} \bar{*} F_{(4)} \wedge F_{(4)} \\ &\quad - \frac{1}{2} e^{-\phi} \bar{*} F_{(3)} \wedge F_{(3)} + \frac{1}{2} dA_{(3)} \wedge dA_{(3)} \wedge A_{(2)}, \end{aligned} \quad (55)$$

where

$$F_{(4)} = dA_{(3)} - dA_{(2)} \wedge \mathcal{A}_{(1)}, \quad F_{(3)} = dA_{(2)}, \quad \mathcal{F}_{(2)} = d\mathcal{A}_{(1)}. \quad (56)$$

(In this subsection, we use a bar where necessary to indicate ten-dimensional quantities.)

The equations of motion derived from the above Lagrangian are

$$\begin{aligned} d\bar{*} d\phi &= \frac{1}{2} e^{-\phi} \bar{*} F_{(3)} \wedge F_{(3)} - \frac{3}{4} e^{\frac{3}{2}\phi} \bar{*} \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} - \frac{1}{4} e^{\frac{1}{2}\phi} \bar{*} F_{(4)} \wedge F_{(4)}, \\ d(e^{\frac{1}{2}\phi} \bar{*} F_{(4)}) &= F_{(4)} \wedge F_{(3)}, \end{aligned}$$

$$\begin{aligned} d(e^{\frac{3}{2}\phi} \bar{*}\mathcal{F}_{(2)}) &= -e^{\frac{1}{2}\phi} \bar{*}F_{(4)} \wedge F_{(3)}, \\ d(e^{-\phi} \bar{*}F_{(3)}) &= \frac{1}{2}F_{(4)} \wedge F_{(4)} - e^{\frac{1}{2}\phi} \bar{*}F_{(4)} \wedge \mathcal{F}_{(2)}. \end{aligned} \quad (57)$$

Note that it is consistent to truncate the theory to the NS-NS sector, namely the subsector comprising the metric, the dilaton and the 3-form field strength. This implies that it is possible also to perform an  $S^3$  reduction of the NS-NS sector alone, which was indeed demonstrated in [16]. On the other hand it is not consistent to truncate the theory to a subsector comprising only the metric, the dilaton and the 4-form field strength, which again is in agreement with the conclusion in [16] that it is not consistent to perform an  $S^4$  reduction on such a subsector. However, as we show in section 5, there *is* a consistent  $S^4$  reduction if we include all the fields of the type IIA theory.

The reduction Ansatz obtained in section (4.1) can now be substituted into the type IIA equations of motion, to verify that it indeed leads to the equations of motion for the  $SO(4)$ -gauged seven-dimensional theory constructed in section 3.

## 5 $S^4$ reduction of type IIA supergravity

We can also derive the Ansatz for the consistent  $S^4$  reduction of type IIA supergravity from the  $S^4$  reduction Ansatz of eleven-dimensional supergravity. In this case we do not need to take any singular limit of the internal 4-sphere, but rather, we extract the “extra” coordinate from the seven-dimensional spacetime of the original eleven-dimensional supergravity reduction Ansatz. The resulting six-dimensional  $SO(5)$  gauged maximal supergravity can be obtained from the Kaluza-Klein reduction of seven-dimensional gauged maximal supergravity on a circle.

We begin, therefore, by making a standard  $S^1$  Kaluza-Klein reduction of the seven-dimensional metric:

$$ds_7^2 = e^{-2\alpha\varphi} ds_6^2 + e^{8\alpha\varphi} (dz + \bar{\mathcal{A}}_{(1)})^2, \quad (58)$$

where  $\alpha = 1/\sqrt{40}$ . With this parameterisation the metric reduction preserves the Einstein frame, and the dilatonic scalar  $\varphi$  has the canonical normalisation for its kinetic term in six dimensions.<sup>7</sup> Substituting (58) into the original metric reduction Ansatz (1), we obtain

$$d\hat{s}_{11}^2 = \Delta^{1/3} e^{-2\alpha\varphi} ds_6^2 + \frac{1}{g^2} \Delta^{-2/3} T_{ij}^{-1} D\mu^i D\mu^j + \Delta^{1/3} e^{8\alpha\varphi} (dz + \bar{\mathcal{A}}_{(1)})^2. \quad (59)$$

---

<sup>7</sup>We use a bar to denote six-dimensional fields, in cases where this is necessary to avoid an ambiguity.

In order to extract the Ansatz for the  $S^4$  reduction of type IIA supergravity, we must first rewrite (59) in the form

$$d\hat{s}_{11}^2 = e^{-\frac{1}{6}\phi} ds_{10}^2 + e^{\frac{4}{3}\phi} (dz + \mathcal{A}_{(1)})^2, \quad (60)$$

which is a canonical  $S^1$  reduction from  $D = 11$  to  $D = 10$ . It is not immediately obvious that this can easily be done, since the Yang-Mills fields  $A_{(1)}^{ij}$  appearing in the covariant differentials  $D\mu^i$  in (59) must themselves be reduced according to standard Kaluza-Klein rules,

$$A_{(1)}^{ij} = \bar{A}_{(1)}^{ij} + \chi^{ij} (dz + \bar{\mathcal{A}}_{(1)}), \quad (61)$$

where  $\bar{A}_{(1)}^{ij}$  are the  $SO(5)$  gauge potentials in six dimensions, and  $\chi^{ij}$  are six-dimensional axions. Thus we have

$$D\mu^i = \bar{D}\mu^i + g \chi^{ij} \mu^j (dz + \bar{\mathcal{A}}_{(1)}), \quad (62)$$

where

$$\bar{D}\mu^i \equiv d\mu^i + g \bar{A}_{(1)}^{ij} \mu^j. \quad (63)$$

This means that the differential  $dz$  actually appears in a much more complicated way in (59) than is apparent at first sight. Nonetheless, we find that one can in fact “miraculously” complete the square, and thereby rewrite (59) in the form of (60).

To present the result, it is useful to make the following definitions:

$$\begin{aligned} \Omega &\equiv \Delta^{1/3} e^{8\alpha\varphi} + \Delta^{-2/3} T_{ij}^{-1} \chi^{ik} \chi^{j\ell} \mu^k \mu^\ell, \\ Z_{ij} &\equiv T_{ij}^{-1} - \Omega^{-1} \Delta^{-2/3} T_{ik}^{-1} T_{j\ell}^{-1} \chi^{km} \chi^{\ell n} \mu^m \mu^n, \end{aligned} \quad (64)$$

In terms of these, we find after some algebra that we can rewrite (59) as

$$ds_{11}^2 = \Delta^{1/3} e^{-2\alpha\varphi} ds_6^2 + \frac{1}{g^2} \Delta^{-2/3} Z_{ij} \bar{D}\mu^i \bar{D}\mu^j + \Omega (dz + \mathcal{A}_{(1)})^2, \quad (65)$$

where the ten-dimensional potential  $\mathcal{A}_{(1)}$  is given in terms of six-dimensional fields by

$$\mathcal{A}_{(1)} = \bar{\mathcal{A}}_{(1)} + \frac{1}{g} \Omega^{-1} \Delta^{-2/3} T_{ij}^{-1} \chi^{jk} \mu^k \bar{D}\mu^i. \quad (66)$$

This is therefore the Kaluza-Klein  $S^4$  reduction Ansatz for the 1-form  $\mathcal{A}_{(1)}$  of the type IIA theory. Comparing (65) with (60), we see that the Kaluza-Klein reduction Ansätze for the metric  $ds_{10}^2$  and dilaton  $\phi$  of the type IIA theory are given by

$$\begin{aligned} ds_{10}^2 &= \Omega^{1/8} \Delta^{1/3} e^{-2\alpha\varphi} ds_6^2 + \frac{1}{g^2} \Omega^{1/8} \Delta^{-2/3} Z_{ij} \bar{D}\mu^i \bar{D}\mu^j, \\ e^{\frac{4}{3}\phi} &= \Omega. \end{aligned} \quad (67)$$

The  $S^4$  reduction Ansatz for the R-R 4-form  $F_{(4)}$  of the type IIA theory is obtained in a similar manner, by first implementing a standard  $S^1$  Kaluza-Klein reduction on the various seven-dimensional fields appearing in the  $S^4$  reduction Ansatz (2) for the eleven-dimensional 4-form  $\hat{F}_{(4)}$ , and then matching this to a standard  $S^1$  reduction of  $\hat{F}_{(4)}$  from  $D = 11$  to  $D = 10$ :

$$\hat{F}_{(4)} = F_{(4)} + F_{(3)} \wedge (dz + \mathcal{A}_{(1)}) . \quad (68)$$

Note that in doing this, it is appropriate to treat the 3-form fields  $S_{(3)}^i$  of the seven-dimensional theory as field strengths for the purpose of the  $S^1$  reduction to  $D = 6$ , *viz.*

$$S_{(3)}^i = \bar{S}_{(3)}^i + \bar{S}_{(2)}^i \wedge (dz + \bar{\mathcal{A}}_{(1)}) . \quad (69)$$

It is worth noting also that this implies that the reduction of the seven-dimensional Hodge duals  $*S_{(3)}^i$  will be given by

$$*S_{(3)}^i = e^{4\alpha\varphi} \bar{*}\bar{S}_{(3)}^i \wedge (dz + \bar{\mathcal{A}}_{(1)}) + e^{-6\alpha\varphi} \bar{*}\bar{S}_{(2)}^i , \quad (70)$$

where  $\bar{*}$  denotes a Hodge dualisation in the six-dimensional metric  $ds_6^2$ .

With these preliminaries, it is now a mechanical, albeit somewhat uninspiring, exercise to make the necessary substitutions into (2), and, by comparing with (68), read off the expressions for  $F_{(4)}$  and  $F_{(3)}$ . These give the Kaluza-Klein  $S^4$  reductions Ansätze for the 4-form and 3-form field strengths of type IIA supergravity. We shall not present the results explicitly here, since they are rather complicated, and are easily written down “by inspection” if required. For these purposes, the following identities are useful:

$$\begin{aligned} (dz + \bar{\mathcal{A}}_{(1)}) &= (dz + \mathcal{A}_{(1)}) - \frac{1}{g} \Omega^{-1} \Delta^{-2/3} T_{ij}^{-1} \chi^{jk} \mu^k \bar{D}\mu^i , \\ D\mu^i &= T_{ij} Z_{jk} \bar{D}\mu^k + g \chi^{ij} \mu^j (dz + \mathcal{A}_{(1)}) , \\ DX_i &= \bar{D}X_i - \Omega^{-1} \Delta^{-2/3} \chi^{ij} X_j T_{kl}^{-1} \chi^{\ell m} \mu^m \bar{D}\mu^k + g \chi^{ij} (dz + \mathcal{A}_{(1)}) , \end{aligned} \quad (71)$$

where in the last line  $X_i$  represents any six-dimensional field in the vector representation of  $SO(5)$ , and the covariant derivative generalises to higher-rank  $SO(5)$  tensors in the obvious way.

If we substitute the  $S^4$  reduction Ansätze given for the ten-dimensional dilaton, metric and 1-form in (67), and (66), together with those for  $F_{(4)}$  and  $F_{(3)}$  as described above, into the equations of motion of type IIA supergravity, we shall obtain a consistent reduction to six dimensions. This six-dimensional theory will be precisely the one that follows by performing an ordinary  $S^1$  Kaluza-Klein reduction on the  $SO(5)$ -gauged maximal supergravity in  $D = 7$ , whose bosonic Lagrangian is given in (39).

It is perhaps worth remarking that the expression (67) for the Kaluza-Klein  $S^4$  reduction of the type IIA supergravity metric illustrates a point that has been observed previously (for example in [7, 10]), namely that the Ansatz becomes much more complicated when axions or pseudoscalars are involved. Although the axions  $\chi^{ij}$  would not be seen in the metric Ansatz in a linearised analysis, they make an appearance in a rather complicated way in the full non-linear Ansatz that we have obtained here, for example in the quantities  $\Omega$  and  $Z_{ij}$  defined in (64). They will also, of course, appear in the Ansätze for the  $F_{(4)}$  and  $F_{(3)}$  field strengths. It may be that the results we are finding here could be useful in other contexts, for providing clues as to how the axionic scalars should appear in the Kaluza-Klein reduction Ansatz.

## 6 Conclusions

In this paper, we have obtained a consistent 3-sphere reduction of type IIA supergravity, in which all the massless  $SO(4)$  gauge bosons associated with the isometry group of the 3-sphere are retained. The resulting seven-dimensional gauged supergravity will, accordingly, be maximally supersymmetric. It is, however, not a theory that admits an  $AdS_7$  vacuum solution, but rather, it allows a domain wall as its “most symmetric” ground state. Since the 3-sphere is isomorphic to  $SU(2)$  our construction can be set in the context of a string propagating in a group-manifold background. However, the reduction of fields that we considered here goes beyond what is customarily included in such cases, since we can retain the entire set of  $SO(4) \sim SU(2)_L \times SU(2)_R$  Yang-Mills fields, and not merely those of either the left-acting or right-acting  $SU(2)$ .

It is perhaps worth emphasising that although we can interpret the  $\mathbb{R} \times S^3$  limit of the  $S^4$  reduction from  $D = 11$  as an  $S^3$  reduction of the type IIA theory, we cannot reverse the roles of the  $\mathbb{R}$  and  $S^3$  factors and interpret the limit as an  $S^3$  reduction of eleven-dimensional supergravity to give an  $SO(4)$ -gauged supergravity in  $D = 8$ , which then undergoes a further reduction to  $D = 7$ . The reason for this is that when the limiting procedure is applied to the  $\mu^i$  coordinates of  $S^4$ , as in (41), the original coordinate  $\mu^0$  is set to zero, and so all fields necessarily become independent of the rescaled coordinate  $\tilde{\mu}^0$  on the  $\mathbb{R}$  factor. This means that the consistent reduction involving  $S^3$  in the limit works only if the fields are all assumed to be independent of the coordinate  $\tilde{\mu}^0$  as well, and so there would be no possibility of extracting an eight-dimensional covariant theory by just considering the  $S^3$  factor in the  $\mathbb{R} \times S^3$  reduction.

The consistent  $S^3$  reduction of type IIA supergravity that we have constructed in this paper represents another element in the accumulating body of examples where “remarkable” Kaluza-Klein sphere reductions exist, even though there is no known group-theoretic explanation for their consistency. What is still lacking is a deeper understanding of why they should work. One might be tempted to think that supersymmetry could provide the key, but this evidently cannot in general be the answer, since there are examples such as the consistent  $S^3$  and  $S^{D-3}$  reductions of the  $D$ -dimensional low-energy limit of the bosonic string (in arbitrary dimension  $D$ ) [16] which are obviously unrelated to supersymmetry.

As we discussed in introduction, we expect further examples of consistent sphere reduction in type IIA and type IIB supergravities. In particular, for non-trivial vacuum NS-NS flux, we expect that it is consistent to reduce both type IIA and type IIB on  $S^3$  and  $S^7$ . For non-trivial vacuum R-R flux, we expect that it is consistent to reduce the type IIA theory on  $S^n$  with  $n = 2, 4, 6, 8$  and for the type IIB theory on  $S^n$  with  $n = 1, 3, 5, 7$ . The resulting maximal gauged supergravities in the lower dimensions in general have domain-walls rather than AdS as vacuum solutions, except in the case  $n = 5$  for type IIB. We constructed two such examples in this paper, namely the  $S^3$  and  $S^4$  reductions of the type IIA theory. These domain-wall supergravities provide useful tools with which to explore the Domain Wall/QFT correspondence.

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